

# Two questions of L. Vaš on $\ast$ -clean rings

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## ABSTRACT

A ring  $R$  with an involution  $\ast$  is called (strongly)  $\ast$ -clean if every element of  $R$  is the sum of a unit and a projection (that commute). All  $\ast$ -clean rings are clean. Vaš [L. Vaš,  $\ast$ -Clean rings; some clean and almost clean Baer  $\ast$ -rings and von Neumann algebras, J. Algebra 324 (12) (2010) 3388-3400] asked whether there exists a  $\ast$ -ring that is clean but not  $\ast$ -clean and whether a unit regular and  $\ast$ -regular ring is strongly  $\ast$ -clean. In this paper, we answer both questions by several examples. Moreover, some characterizations of unit regular and  $\ast$ -regular rings are provided.

*Keywords:*  $\ast$ -Clean ring; Strongly  $\ast$ -Clean ring;  $\ast$ -Regular ring; Strongly clean ring; Unit regular ring

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## 1. INTRODUCTION

Rings in which every element is the product of a unit and an idempotent are said to be *unit regular*, and have been extensively studied. As a result due to Camillo and Khurana [2], every element of a unit regular ring can also be written as the sum of a unit and an idempotent. Recall that an element  $a$  of a ring  $R$  is *clean* if  $a = e + u$  where  $e^2 = e \in R$  and  $u$  is a unit of  $R$ , and  $R$  is called *clean* if every element of  $R$  is clean. Clean rings were introduced by Nicholson [5] in relation to exchange rings. In 1999, Nicholson [6] called an element of a ring  $R$  *strongly clean* if it is the sum of a unit and an idempotent that commute with each other, and  $R$  is *strongly clean* if each of its elements is strongly clean. Clearly, a strongly clean ring is clean, and the converse holds

for abelian rings (that is, all idempotents in the ring are central). Local rings and strongly  $\pi$ -regular rings are well-known examples of strongly clean rings.

A ring  $R$  is a *\*-ring* (or *ring with involution*) if there exists an operation  $*$  :  $R \rightarrow R$  such that for all  $x, y \in R$

$$(x + y)^* = x^* + y^*, \quad (xy)^* = y^*x^*, \quad \text{and} \quad (x^*)^* = x.$$

Clearly,  $1^* = 1$  and  $0^* = 0$  in any *\*-ring*. An element  $p$  of a *\*-ring*  $R$  is said to be a *projection* if  $p^2 = p = p^*$ . Recently, Vaš [7] introduced the concepts of a *\*-clean ring* and a *strongly \*-clean ring*. Following [7], an element of a *\*-ring*  $R$  is called (*strongly*) *\*-clean* if it can be expressed as the sum of a unit and a projection (that commute), and  $R$  is called *\*-clean* (*resp.*, *strongly \*-clean*) in case all of its elements are *\*-clean* (*resp.*, *strongly \*-clean*). Strongly *\*-clean* rings are strongly clean and *\*-clean*, and *\*-clean* rings are clean, but it is a question that whether there is an example of a *\*-ring* that is clean but not *\*-clean* [7]. According to [1, Proposition 3, p. 229], a *\*-ring*  $R$  is *\*-regular* if one of the following equivalent conditions hold: (1)  $R$  is a (von Neumann) regular and Rickart *\*-ring* ( i.e., the right annihilator of each element is generated by a projection); (2)  $R$  is regular and the involution is proper (that is,  $x^*x = 0$  implies  $x = 0$  for any  $x \in R$ ); (3) for every  $x$  in  $R$  there exists a projection  $p$  such that  $xR = pR$ . It was shown in [7] that every *\*-abelian* (i.e., a *\*-ring* in which every projection is central) and *\*-regular ring* is strongly *\*-clean*. Vaš asked whether a unit regular and *\*-regular ring* is also strongly *\*-clean*.

In this paper, examples of *\*-rings* are provided to answer both questions of Vaš. Some properties of (strongly) *\*-clean rings* are investigated. In particular, we show that in *\*-rings* setting, a strongly clean ring is strongly *\*-clean* iff the set of all projections coincides with the set of all idempotents. Several characterizations of unit regular and *\*-regular rings* are given.

Rings considered in this paper are associative with unity. The notation  $*$  denotes an involution of a given ring. The set of all idempotents, all projections and all units of a ring  $R$  are denoted by  $Id(R)$ ,  $P(R)$  and  $U(R)$ , respectively. The symbol  $l(X)$  (*resp.*,  $r(X)$ ) stands for the left (*resp.*, right) annihilator of a subset  $X \subseteq R$ . We write  $M_n(R)$  for the ring of all  $n \times n$  matrices over  $R$ .

## 2. Main Results

Let  $R$  be a *\*-ring* and  $p \in P(R)$ . The involution  $*$  of  $R$  is inherited naturally to the corner ring  $pRp$ .

**Theorem 1.** *Let  $R$  be a  $*$ -ring and  $p \in P(R)$ . Then  $a \in pRp$  is strongly  $*$ -clean in  $R$  if and only if  $a$  is strongly  $*$ -clean in  $pRp$ .*

*Proof.* Assume that  $a = e + u$  is strongly  $*$ -clean in  $pRp$  with  $e \in P(pRp)$ ,  $u \in U(pRp)$  and  $ue = eu$ . Let  $f = e + (1 - p)$  and  $v = u - (1 - p)$ . Then  $f \in P(R)$ ,  $v \in U(R)$ , and  $f$  commutes with  $v$ . So  $a = f + v$  is strongly  $*$ -clean in  $R$ .

Conversely, suppose that  $a \in pRp$  is strongly  $*$ -clean in  $R$ . Let  $a = e + u$  with  $e \in P(R)$ ,  $u \in U(R)$  and  $ue = eu$ . Since  $a = pap$ ,  $1 - p \in r(a) \cap l(a)$ . By [6, Theorem 2],  $r(a) \subseteq eR$  and  $l(a) \subseteq Re$ . So we have  $1 - p \in eR \cap Re = eRe$ , and then  $(1 - p)e = e(1 - p)$ , i.e.,  $ep = pe$ . This implies that  $pep \in Id(pRp)$ . Note that both  $e$  and  $p$  are in  $P(R)$ . Thus  $pep \in P(pRp)$ . Since  $ap = pa$  and  $ep = pe$ ,  $up = pu$ . It follows that  $pup \in U(pRp)$ , and  $pep$  commutes with  $pup$ . Therefore,  $a = pep + pup$  is a strongly  $*$ -clean expression in  $pRp$ .  $\square$

**Corollary 2.** *If  $R$  is a strongly  $*$ -clean ring, then  $pRp$  is strongly  $*$ -clean for any  $p \in P(R)$ .*

The following result, which reveals the relationship between strong  $*$ -cleanness and strong cleanness, is crucial for constructing a counter-example of a  $*$ -ring that is strongly clean but not strongly  $*$ -clean.

**Theorem 3.** *Let  $R$  be a  $*$ -ring. Then  $R$  is strongly  $*$ -clean if and only if  $R$  is strongly clean and  $P(R) = Id(R)$ .*

*Proof.* Suppose that  $R$  is strongly  $*$ -clean. The strong cleanness of  $R$  is clear. For any  $e^2 = e \in R$ , we have  $e = p + u$  where  $p \in P(R)$ ,  $u \in U(R)$  and  $e, p$  and  $u$  commute with each other. If  $p = 0$  then  $e = 1$ , and if  $p = 1$  then  $e = 0$ . Notice that both 0 and 1 are contained in  $P(R)$ . We may assume that  $p \neq 0$  and  $p \neq 1$ . Then  $pRp$  and  $(1 - p)R(1 - p)$  are nonzero  $*$ -rings. Now, multiplying  $e = p + u$  by  $p$  yields  $ep = p + up$ . It follows that  $-up = p - ep = (1 - e)p \in U(pRp) \cap Id(pRp) = \{p\}$ . Thus  $ep = 0$ . Analogously, by multiplying  $1 - p$  on both sides of  $e = p + u$  we obtain that  $e(1 - p) = u(1 - p) \in U[(1 - p)R(1 - p)] \cap Id[(1 - p)R(1 - p)] = \{1 - p\}$ . So one has  $e(1 - p) = 1 - p$ . Since  $ep = 0$ ,  $e = 1 - p$ . Clearly,  $e = e^*$ . This proves that  $Id(R) = P(R)$ . The other direction is trivial.  $\square$

Due to [7], if  $R$  is a  $*$ -ring, the ring  $M_n(R)$  has a natural involution inherited from  $R$ : if  $A = (a_{ij}) \in M_n(R)$ ,  $A^*$  is the transpose of  $(a_{ij}^*)$ . Thus  $M_n(R)$  is also a  $*$ -ring. It was shown that  $M_n(R)$  is a  $*$ -clean ring whenever  $R$  is  $*$ -clean [7, Proposition 4]. By Theorem 3, we have the following result.

**Corollary 4.** *Let  $R$  be a  $*$ -ring. Then  $M_n(R)$  is not strongly  $*$ -clean for  $n \geq 2$ .*

Note that a local ring  $R$  with an involution  $*$  is always strongly  $*$ -clean. So  $M_n(R)$  is  $*$ -clean, but it is not strongly  $*$ -clean when  $n \geq 2$ . By [3, Corollary 1.9], there exists a commutative local ring  $R$  such that  $M_2(R)$  is not strongly clean. Vaš [7] asked whether there is an example of a  $*$ -ring that is clean but not  $*$ -clean. We answer this question affirmatively by the following example.

**Example 5.** *Let  $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , where  $\mathbb{Z}_2$  is the ring of integers modulo 2. It is clear that  $R$  is a boolean ring. Thus  $R$  is strongly clean and  $R = Id(R)$ . Define a map  $*$  :  $R \rightarrow R$  by  $(a, b)^* = (b, a)$ . Since  $R$  is commutative, it is easy to check that  $*$  is an involution of  $R$ . Note that  $P(R) = \{(0, 0), (1, 1)\} \neq Id(R)$ . In view of Theorem 3,  $R$  is not strongly  $*$ -clean, and thus not  $*$ -clean because  $R$  is commutative.*

**Remark 6.** *Example 5 shows that strongly clean  $*$ -rings need not be  $*$ -clean. The following implications hold (for the class of  $*$ -rings) :*

$$\begin{array}{ccc} \text{strongly } * \text{-clean ring} & \Longrightarrow & * \text{-clean ring} \\ \Downarrow & & \Downarrow \\ \text{strongly clean ring} & \Longrightarrow & \text{clean ring} \end{array}$$

*In the table above, each of the implications is irreversible, and there are no other implications between these rings.*

Recall that a ring  $R$  is *right P-injective* if  $lr(a) = Ra$  for each  $a \in R$ . Regular rings are clearly right P-injective.

**Proposition 7.** *Let  $R$  be a  $*$ -ring. Then the following are equivalent:*

- (1)  *$R$  is regular and the involution is proper (i.e.,  $R$  is  $*$ -regular).*
- (2)  *$R$  is right P-injective and the involution is proper.*
- (3) *For every  $a \in R$ ,  $Ra = Ra^*a$ .*

*Proof.* (1)  $\Rightarrow$  (2) is clear.

(2)  $\Rightarrow$  (3). Given any  $a \in R$ . Let  $y \in r(a^*a)$ . We have  $a^*ay = 0$ , and it follows that  $0 = y^*a^*ay = (ay)^*(ay)$ . Since the involution is proper,  $ay = 0$ , i.e.,  $y \in r(a)$ . Thus,  $r(a^*a) = r(a)$ . By the right P-injectivity of  $R$ , we obtain  $Ra = lr(a) = lr(a^*a) = Ra^*a$ .

(3)  $\Rightarrow$  (1). For any  $a \in R$ , there exists  $t \in R$  such that  $a = ta^*a$ . Then  $at^*a = (ta^*a)t^*a = t(a^*at^*)a = t(ta^*a)^*a = ta^*a = a$ . This proves that  $R$  is a regular ring. We finish by proving that the involution is proper. Let  $x^*x = 0$  with  $x \in R$ . By (3),  $Rx = Rx^*x = 0$ , so  $x = 0$ , as desired.  $\square$

A ring  $R$  is called *strongly regular* if it is an abelian regular ring, or equivalently, for any  $a \in R$ ,  $a = eu = ue$  for  $e \in Id(R)$  and  $u \in U(R)$  [6].

**Proposition 8.** *Let  $R$  be a  $*$ -ring. Then the following are equivalent:*

- (1)  *$R$  is strongly regular and the involution is proper.*
- (2)  *$R$  is strongly regular and  $P(R) = Id(R)$ .*
- (3)  *$R$  is  $*$ -abelian and for every  $a \in R$ ,  $a = p + u$  with  $aR \cap pR = 0$  where  $p \in P(R)$  and  $u \in U(R)$ .*
- (4) *For every  $a \in R$ ,  $a = pu = up$  for some  $p \in P(R)$  and  $u \in U(R)$ .*

*Proof.* (1)  $\Rightarrow$  (2). In view of Proposition 7,  $R$  is  $*$ -regular. Since  $R$  is also abelian, by [7, Lemma 3]  $P(R) = Id(R)$ .

(2)  $\Rightarrow$  (3). Every abelian  $*$ -ring is  $*$ -abelian; and the rest follows from [2, Theorem 1].

(3)  $\Rightarrow$  (4). Let  $a \in R$ . Then there exists  $1 - p \in P(R)$  and  $u \in U(R)$  such that  $a = (1 - p) + u$  and  $aR \cap (1 - p)R = 0$ . Since  $R$  is  $*$ -abelian,  $a(1 - p) \in aR \cap R(1 - p) = aR \cap (1 - p)R = 0$ . Hence,  $a = ap = up = pu$ .

(4)  $\Rightarrow$  (1). The strong regularity of  $R$  is clear. We assume that  $x^*x = 0$  for  $x \in R$ . Then  $x = pu = up$  for some  $p \in P(R)$  and  $u \in U(R)$ . Obviously,  $0 = x^*x = u^*pu$ . Thus  $p = 0$ , and so  $x = 0$ . Therefore, the involution  $*$  of  $R$  is proper.  $\square$

A ring  $R$  is said to *have stable range 1* provided that whenever  $aR + bR = R$  for any  $a, b \in R$ , there exists  $t \in R$  such that  $a + bt$  is a unit of  $R$ . Next we give some characterizations of unit regular and  $*$ -regular rings.

**Theorem 9.** *Let  $R$  be a  $*$ -ring. Then the following are equivalent:*

- (1)  *$R$  is unit regular and the involution is proper.*
- (2)  *$R$  is unit regular and  $*$ -regular.*
- (3) *For every  $a \in R$ ,  $a = pu$  where  $p \in P(R)$  and  $u \in U(R)$ .*
- (4) *For every  $a \in R$ ,  $a = vq$  where  $q \in P(R)$  and  $v \in U(R)$ .*

*Proof.* (1)  $\Rightarrow$  (2) follows by Proposition 7.

(2)  $\Rightarrow$  (3). For any  $a \in R$ , there exists  $e \in Id(R)$  and  $w \in U(R)$  such that  $a = ew$ . Since  $R$  is  $*$ -regular,  $eR = pR$  for some projection  $p \in R$ . Thus  $e = pe$ . Note that  $eR + (1 - p)R = R$ . In view of [4, Proposition 4.12],  $R$  has stable range 1. So there exists  $t \in R$  satisfying  $e + (1 - p)t = v \in U(R)$ . Clearly,  $pe = pv$ . It follows that  $e = pe = pv$  and  $a = ew = p(vw)$ . Write  $u = vw$ . Then  $u \in U(R)$  and  $a = pu$ .

(3)  $\Rightarrow$  (4). Given  $a \in R$ , let  $b = a^*$ . By hypothesis,  $b = pu$  with  $p \in P(R)$  and  $u \in U(R)$ . Then  $a = b^* = u^*p$ . Taking  $v = u^*$  and  $q = p$ , it follows that  $v \in U(R)$ ,  $q \in P(R)$  and  $a = vq$ .

(4)  $\Rightarrow$  (1).  $R$  is clearly unit regular, so it suffices to show that the involution is proper. Let  $a \in R$  with  $a^*a = 0$ . By (4),  $a^* = vq$  for some  $v \in U(R)$  and  $q \in P(R)$ . Thus  $0 = a^*a = (vq)(qv^*) = vqv^*$ . So we have  $q = 0$ , which implies that  $a = 0$ . We obtain the required result.  $\square$

**Definition 10.** A  $*$ -ring  $R$  is called  $*$ -unit regular if  $R$  satisfies the conditions in Theorem 9.

**Proposition 11.** Let  $R$  be a  $*$ -ring and  $n$  a positive integer. The following are equivalent:

- (1)  $M_n(R)$  is  $*$ -unit regular.
- (2)  $R$  is unit regular and  $a_1^*a_1 + a_2^*a_2 + \cdots + a_n^*a_n = 0$  implies  $a_i = 0$  for all  $i$ .

*Proof.* (1)  $\Rightarrow$  (2). Since  $M_n(R)$  is  $*$ -unit regular, it is unit regular. By [4, Corollary 4.7],  $R$  is unit regular. Suppose that  $a_1^*a_1 + a_2^*a_2 + \cdots + a_n^*a_n = 0$  for some  $a_i \in R$ . Let  $A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix} \in M_n(R)$ . Then  $A^*A = 0$ . Since the involution  $*$  of  $M_n(R)$  is proper,  $A = 0$ . Thus  $a_1 = a_2 = \cdots = a_n = 0$ .

(2)  $\Rightarrow$  (1). As  $R$  is a unit regular ring, so is  $M_n(R)$  by [4, Corollary 4.7]. The remaining proof is to show that the involution  $*$  of  $M_n(R)$  is proper. Let  $A = (a_{ij}) \in M_n(R)$  with  $A^*A = 0$ . We have

$$a_{1j}^*a_{1j} + a_{2j}^*a_{2j} + \cdots + a_{nj}^*a_{nj} = 0$$

where  $j = 1, \dots, n$ . Then, the hypothesis implies that  $a_{ij} = 0$  for all  $i, j$ . Thus we have  $A = 0$ , and the proof is complete.  $\square$

Based on Proposition 11, we have the following examples.

**Example 12.** Let  $\mathbb{R}$ ,  $\mathbb{C}$  be the fields of real numbers and complex numbers, respectively. Clearly, both  $\mathbb{R}$  and  $\mathbb{C}$  are unit regular.

- (1) Let  $*$  :  $\mathbb{R} \rightarrow \mathbb{R}$  be an involution defined by  $x \mapsto x$ . Then  $M_n(\mathbb{R})$  is  $*$ -unit regular.
- (2) Define an involution  $*$  of the ring  $\mathbb{C}$  by  $x \mapsto \bar{x}$ , where  $\bar{x}$  is the conjugation of  $x$ . It can be directly checked that  $M_n(\mathbb{C})$  is  $*$ -unit regular.
- (3) Let  $R = \mathbb{R} \times \mathbb{R}$  be a ring with the usual addition and multiplication. An involution  $*$  of  $R$  is given by  $x \mapsto x$ . Then  $R$  is unit regular and  $M_n(R)$  is  $*$ -unit regular.

(4) Let  $*$  :  $x \mapsto x$  be an involution of  $\mathbb{Z}_2$ . Then  $M_2(\mathbb{Z}_2)$  is not  $*$ -unit regular because  $1^* \cdot 1 + 1^* \cdot 1 = 0$  but  $1 \neq 0$ .

In [6], Nicholson asked whether a unit regular ring is strongly clean, it is still an open problem. Vaš [7] raised a question if a unit regular and  $*$ -regular ring is strongly  $*$ -clean, which is equivalent to asking whether a  $*$ -unit regular ring is strongly  $*$ -clean. Here we give a negative answer.

**Example 13.** Let  $R$  be a  $*$ -ring as defined in Example 12(1), (2) and (3). Then  $M_2(R)$  is  $*$ -unit regular. Nevertheless, by Corollary 4  $M_2(R)$  is not strongly  $*$ -clean.

According to Example 12(4), the matrix ring of a  $*$ -unit regular ring need not be  $*$ -unit regular. However, we have the following result for its corner ring.

**Proposition 14.** If  $R$  is a  $*$ -unit regular ring, then  $pRp$  is  $*$ -unit regular for every  $p \in P(R)$ .

*Proof.* The ring  $R$  is unit regular, by [4, Corollary 4.7]  $pRp$  is unit regular as well where  $p \in P(R)$ . Let  $a \in pRp$  ( $\subseteq R$ ) with  $a^*a = 0$ . Since  $R$  is  $*$ -unit regular, we deduce that  $a = 0$ . So the involution in  $pRp$  is proper. Thus  $pRp$  is  $*$ -unit regular by Theorem 9.  $\square$

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